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# A New Intersection Model and Improved Algorithms for Tolerance Graphs

George B. Mertzios*, Ignasi Sau**, and Shmuel Zaks***

**Abstract.** Tolerance graphs model interval relations in such a way that intervals can tolerate a certain degree of overlap without being in conflict. This class of graphs, which generalizes in a natural way both interval and permutation graphs, has attracted many research efforts since their introduction in [10], as it finds many important applications in constraint-based temporal reasoning, resource allocation, and scheduling problems, among others. In this article we propose the first non-trivial intersection model for general tolerance graphs, given by three-dimensional parallelepipeds, which extends the widely known intersection model of parallelograms in the plane that characterizes the class of bounded tolerance graphs. Apart from being important on its own, this new representation also enables us to improve the time complexity of three problems on tolerance graphs. Namely, we present optimal $\mathcal{O}(n \log n)$ algorithms for computing a minimum coloring and a maximum clique, and an $\mathcal{O}(n^2)$ algorithm for computing a maximum weight independent set in a tolerance graph with $n$ vertices, thus improving the best known running times $\mathcal{O}(n^2)$ and $\mathcal{O}(n^3)$ for these problems, respectively.

**Keywords:** Tolerance graphs, parallelogram graphs, intersection model, minimum coloring, maximum clique, maximum weight independent set.

## 1 Introduction

A graph $G = (V, E)$ on $n$ vertices is a *tolerance graph* if there is a set $I = \{I_i \mid i = 1, \ldots, n\}$ of closed intervals on the real line and a set $T = \{t_i \mid i = 1, \ldots, n\}$ of positive real numbers, called *tolerances*, such that for any two vertices $v_i, v_j \in V$, $v_iv_j \in E$ if and only if $|I_i \cap I_j| \geq \min\{t_i, t_j\}$, where $|I|$ denotes the length of the interval $I$. These sets of intervals and tolerances form a *tolerance representation* of $G$. If $G$ has a tolerance representation such that $t_i \leq |I_i|$ for $i = 1, \ldots, n$, then $G$ is called a *bounded tolerance graph* and its representation is a *bounded tolerance representation*.

Tolerance graphs were introduced in [10], mainly motivated by the need to solve scheduling problems in which resources that would be normally used exclusively, like rooms or vehicles, can tolerate some sharing among users. Since then, tolerance graphs have been widely studied in the literature [1,2,5,11,12,15,19,22], as they naturally generalize both interval graphs (when all tolerances are equal) and permutation graphs (when $|I_i| = t_i$ for $i = 1, \ldots, n$) [10]. For more details, see [13].

**Notation.** All the graphs considered in this paper are finite, simple, and undirected. Given a graph $G = (V, E)$, we denote by $n$ the cardinality of $V$. An edge between vertices $u$ and $v$ is denoted by $uv$, and in this case vertices $u$ and $v$ are said to be *adjacent*. $\overline{G}$ denotes the *complement* of $G$, i.e. $\overline{G} = (V, \overline{E})$, where $uv \in \overline{E}$ if and only if $uv \notin E$. Given a subset of vertices $S \subseteq V$, the graph $G[S]$ denotes the graph *induced* by the vertices in $S$, i.e. $G[S] = (S, F)$, where for any two vertices $u, v \in S$, $uv \in F$ if and only if $uv \in E$. A subset $S \subseteq V$ is an *independent set* in $G$ if the graph $G[S]$ has no edges. For a subset $K \subseteq V$, the induced subgraph $G[K]$ is a *complete subgraph* of $G$, or a *clique*, if each two of its vertices are adjacent (equivalently, $K$ is

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an independent set in $\overline{G}$). The maximum cardinality of a clique in $G$ is denoted by $\omega(G)$ and is termed the *clique number* of $G$. A *proper coloring* of $G$ is an assignment of different colors to adjacent vertices, which results in a partition of $V$ into independent sets. The minimum number of colors for which there exists a proper coloring is denoted by $\chi(G)$ and is termed the *chromatic number* of $G$. A partition of $V$ into $\chi(G)$ independent sets is a *minimum coloring* of $G$.

**Motivation and previous work.** Besides generalizing interval and permutation graphs in a natural way, the class of tolerance graphs has other important subclasses and superclasses. Let us briefly survey some of them.

Given a class of graphs $\mathcal{G}$, a graph $G$ is a *probe graph* of $\mathcal{G}$ if its vertices can be partitioned into a set $\mathbb{P}$ of probes and an independent set $\mathbb{N}$ of nonprobes, such that $G$ can be extended to a graph of $\mathcal{G}$ by adding edges between certain nonprobes. Probe interval graphs, which are a superclass of interval graphs, have proved to be useful in the Human Genome Project [23]. Tolerance graphs are also a superclass of interval probe graphs [12].

A graph is *perfect* if the chromatic number of every induced subgraph equals the clique number of that subgraph. Perfect graphs include many important families of graphs, and serve to unify results relating colorings and cliques in those families. For instance, in all perfect graphs, the graph coloring problem, maximum clique problem, and maximum independent set problem can all be solved in polynomial time using the Ellipsoid method [14]. Since tolerance graphs were shown to be perfect [11], there exist polynomial time algorithms for these problems. However, these algorithms are not very efficient and therefore, as it happens for most known subclasses of perfect graphs, it makes sense to devise specific fast algorithms for these problems on tolerance graphs.

A *comparability* graph is a graph which can be transitively oriented. A *co-comparability* graph is a graph whose complement is a comparability graph. Bounded tolerance graphs are co-comparability graphs [10], and therefore all known polynomial time algorithms for co-comparability graphs apply to bounded tolerance graphs. This is one of the main reasons why for many problems the existing algorithms have better running time in bounded tolerance graphs than in general tolerance graphs.

A graph $G = (V, E)$ is the *intersection graph* of a family $F = \{S_1, \ldots, S_n\}$ of distinct nonempty subsets of a set $S$ if there exists a bijection $\mu : V \to F$ such that for any two distinct vertices $u, v \in V$, $uv \in E$ if and only if $\mu(u) \cap \mu(v) \neq \emptyset$. In that case, we say that $F$ is an *intersection model* of $G$. It is easy to see that each graph has a trivial intersection model based on adjacency relations [21]. Some intersection models provide a natural and intuitive understanding of the structure of a class of graphs, and turn out to be very helpful to find efficient algorithms to solve optimization problems [21]. Therefore, it is of great importance to establish non-trivial intersection models for families of graphs. A graph $G$ on $n$ vertices is a *parallelogram graph* if we can fix two parallel lines $L_1$ and $L_2$, and for each vertex $v_i \in V(G)$ we can assign a parallelogram $\overline{P}_i$ with parallel sides along $L_1$ and $L_2$ so that $G$ is the intersection graph of $\{\overline{P}_i \mid i = 1, \ldots, n\}$. It was proved in [1, 20] that a graph is a bounded tolerance graph if and only if it is a parallelogram graph. This characterization provides a useful way to think about bounded tolerance graphs. However, this intersection model cannot cope with general tolerance graphs, in which the tolerance of an interval can be greater than its length.

**Our contribution.** In this article we present the first non-trivial intersection model for general tolerance graphs, which generalizes the widely known parallelogram representation of bounded tolerance graphs. The main idea is to exploit the third dimension to capture the information given by unbounded tolerances, and as a result parallelograms are replaced with parallelepipeds. The proposed intersection model is very intuitive and can be efficiently con-

structed from a tolerance representation (actually, we show that it can be constructed in linear time).

Apart from being important on its own, this new representation proves to be a powerful tool for designing efficient algorithms for general tolerance graphs. Indeed, using our intersection model we improve the best existing running times of three problems on tolerance graphs. We present algorithms to find a minimum coloring and a maximum clique in $\mathcal{O}(n \log n)$ time, which is optimal (as discussed in Section 3.4). The best existing algorithm was $\mathcal{O}(n^2)$ [12,13]. We also present an algorithm to find a maximum weight independent set in $\mathcal{O}(n^2)$ time, whereas the best known algorithm was $\mathcal{O}(n^3)$ [13]. We note that [22] proposes an $\mathcal{O}(n^2 \log n)$ algorithm to find a maximum *cardinality* independent set on a general tolerance graph, and that [13] refers to an algorithm transmitted by personal communication with running time $\mathcal{O}(n^2 \log n)$ to find a maximum weight independent set on a general tolerance graph; to the best of our knowledge, this algorithm has not been published.

It is important to note that the complexity of recognizing bounded and general tolerance graphs is a challenging open problem [3,13,22], and this is the reason why we assume throughout this paper that along with the input tolerance graph we are also given a tolerance representation of it. The only "positive" result in the literature concerning recognition of tolerance graphs is a linear time algorithm for the recognition of bipartite tolerance graphs [3].

Nevertheless, it was shown in [15] that every tolerance graph has a polynomial sized tolerance representation, and hence tolerance graphs recognition is in the class NP. There exist other graph classes closely related to tolerance graphs. If in the definition of tolerance graphs we replace the operation "min" between tolerances with "+", we obtain the class of *sum-tolerance graphs* [17], and if we replace it with "max" we obtain the class of *max-tolerance graphs*. Max-tolerance graphs recognition is known to be NP-hard [18].

**Organization of the paper.** We provide the new intersection model of general tolerance graphs in Section 2. In Section 3 we present a canonical representation of tolerance graphs, and then show how it can be used in order to obtain optimal $\mathcal{O}(n \log n)$ algorithms for finding a minimum coloring and a maximum clique in a tolerance graph. In Section 4 we present an $\mathcal{O}(n^2)$ algorithm for finding a maximum weight independent set. Finally, Section 5 is devoted to conclusions and open problems.

## 2 A New Intersection Model for Tolerance Graphs

One of the most natural representations of bounded tolerance graphs is given by parallelograms between two parallel lines in the Euclidean plane [1, 13, 20]. In this section we extend this representation to a three-dimensional representation of general tolerance graphs.

Given a tolerance graph $G = (V, E)$ along with a tolerance representation of it, recall that vertex $v_i \in V$ corresponds to an interval $I_i = [a_i, b_i]$ on the real line with a tolerance $t_i \geq 0$. W.l.o.g. we may assume that $t_i > 0$ for every vertex $v_i$ [13].

**Definition 1.** *Given a tolerance representation of a tolerance graph $G = (V, E)$, vertex $v_i$ is* bounded *if $t_i \leq |I_i|$. Otherwise, $v_i$ is* unbounded. *$V_B$ and $V_U$ are the sets of bounded and unbounded vertices in $V$, respectively. Clearly $V = V_B \cup V_U$.*

We can also assume w.l.o.g. that $t_i = \infty$ for any unbounded vertex $v_i$, since if $v_i$ is unbounded, then the intersection of any other interval with $I_i$ is strictly smaller than $t_i$. Let $L_1$ and $L_2$ be two parallel lines at distance 1 in the Euclidean plane.

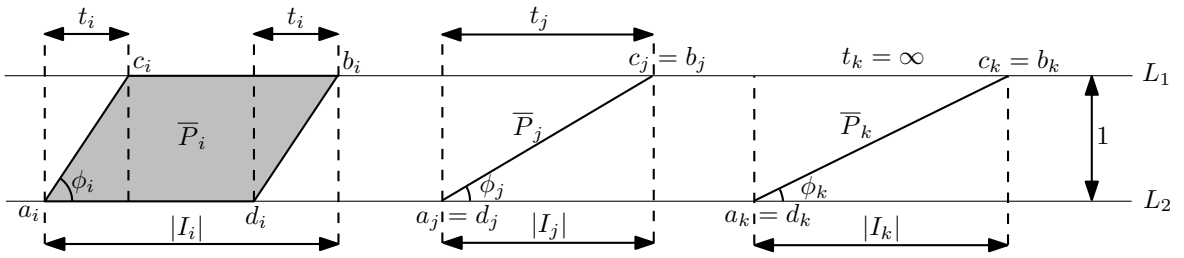


**Fig. 1.** Parallelograms $\overline{P}_i$ and $\overline{P}_j$ correspond to bounded vertices $v_i$ and $v_j$, respectively, whereas $\overline{P}_k$ corresponds to an unbounded vertex $v_k$.

**Definition 2.** *Given an interval $I_i = [a_i, b_i]$ with tolerance $t_i$, $\overline{P}_i$ is the parallelogram defined by the points $c_i, b_i$ in $L_1$ and $a_i, d_i$ in $L_2$, where $c_i = \min\{b_i, a_i + t_i\}$ and $d_i = \max\{a_i, b_i - t_i\}$. The* slope $\phi_i$ *of $\overline{P}_i$ is $\phi_i = \arctan\left(\frac{1}{c_i - a_i}\right)$.*

An example is depicted in Figure 1, where $\overline{P}_i$ and $\overline{P}_j$ correspond to bounded vertices $v_i$ and $v_j$, and $\overline{P}_k$ corresponds to an unbounded vertex $v_k$. Observe that when vertex $v_i$ is bounded, the values $c_i$ and $d_i$ coincide with the *tolerance points* defined in [7,13,16], and $\phi_i = \arctan\left(\frac{1}{t_i}\right)$. On the other hand, when vertex $v_i$ is unbounded, the values $c_i$ and $d_i$ coincide with the endpoints $b_i$ and $a_i$ of $I_i$, respectively, and $\phi_i = \arctan\left(\frac{1}{|I_i|}\right)$. Observe also that in both cases $t_i = b_i - a_i$ and $t_i = \infty$, parallelogram $\overline{P}_i$ is reduced to a line segment (c.f. $\overline{P}_j$ and $\overline{P}_k$ in Figure 1). Since $t_i > 0$ for every vertex $v_i$, it follows that $0 < \phi_i < \frac{\pi}{2}$. Furthermore, we can assume w.l.o.g. that all points $a_i, b_i, c_i, d_i$ and all slopes $\phi_i$ are distinct [7, 13, 16].

**Observation 1** *Let $v_i \in V_U, v_j \in V_B$. Then $|I_i| < t_j$ if and only if $\phi_i > \phi_j$.*

We are ready to give the main definition of this article.

**Definition 3.** *Let $G = (V, E)$ be a tolerance graph with a tolerance representation $\{I_i = [a_i, b_i], t_i \mid i = 1, \ldots, n\}$. For every $i = 1 \ldots, n$, $P_i$ is the parallelepiped in $\mathbb{R}^3$ defined as follows:*

*(a) If $t_i \leq b_i - a_i$ (that is, $v_i$ is bounded), then $P_i = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \overline{P}_i,\ 0 \leq z \leq \phi_i\}$.*
*(b) If $t_i > b_i - a_i$ ($v_i$ is unbounded), then $P_i = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \overline{P}_i,\ z = \phi_i\}$.*

*The set of parallelepipeds $\{P_i \mid i = 1, \ldots, n\}$ is a* parallelepiped representation *of $G$.*

Observe that for each interval $I_i$, the parallelogram $\overline{P}_i$ of Definition 2 (see also Figure 1) coincides with the projection of the parallelepiped $P_i$ on the plane $z = 0$. An example of the construction of these parallelepipeds is given in Figure 2, where a set of eight intervals with their associated tolerances is given in Figure 2(a). The corresponding tolerance graph $G$ is depicted in Figure 2(b), while the parallelepiped representation is illustrated in Figure 2(c). In the case $t_i < b_i - a_i$, the parallelepiped $P_i$ is three-dimensional, c.f. $P_1, P_3$, and $P_5$, while in the border case $t_i = b_i - a_i$ it degenerates to a two-dimensional rectangle, c.f. $P_7$. In these two cases, each $P_i$ corresponds to a bounded vertex $v_i$. In the remaining case $t_i = \infty$ (that is, $v_i$ is unbounded), the parallelepiped $P_i$ degenerates to a one-dimensional line segment above plane $z = 0$, c.f. $P_2, P_4, P_6$, and $P_8$.

We prove now that these parallelepipeds form a three-dimensional intersection model for the class of tolerance graphs (namely, that every tolerance graph $G$ can be viewed as the intersection graph of the corresponding parallelepipeds $P_i$).

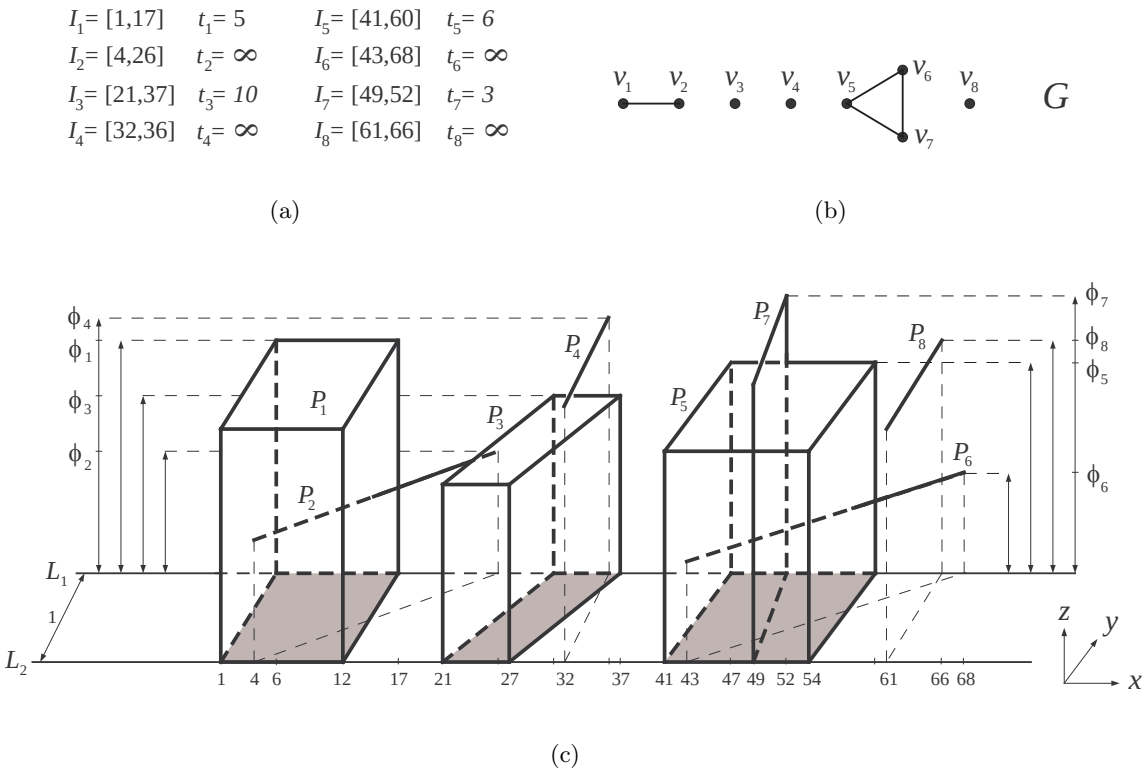


**Fig. 2.** The intersection model for tolerance graphs: (a) a set of intervals $I_i = [a_i, b_i]$ and tolerances $t_i$, $i = 1, \ldots, 8$, (b) the corresponding tolerance graph $G$ and (c) a parallelepiped representation of $G$.

**Theorem 1.** *Let $G = (V, E)$ be a tolerance graph with a tolerance representation $\{I_i = [a_i, b_i], t_i \mid i = 1, \ldots, n\}$. Then for every $i \neq j$, $v_i v_j \in E$ if and only if $P_i \cap P_j \neq \emptyset$.*

**Proof.** We distinguish three cases according to whether vertices $v_i$ and $v_j$ are bounded or unbounded:

(a) Both vertices are bounded, that is $t_i \leq b_i - a_i$ and $t_j \leq b_j - a_j$. It follows from [13] that $v_i v_j \in E(G)$ if and only if $\overline{P}_i \cap \overline{P}_j \neq \emptyset$. However, due to the definition of the parallelepipeds $P_i$ and $P_j$, in this case $P_i \cap P_j \neq \emptyset$ if and only if $\overline{P}_i \cap \overline{P}_j \neq \emptyset$ (c.f. $P_1$ and $P_3$, or $P_5$ and $P_7$, in Figure 2).

(b) Both vertices are unbounded, that is $t_i = t_j = \infty$. Since no two unbounded vertices are adjacent, $v_i v_j \notin E(G)$. On the other hand, the line segments $P_i$ and $P_j$ lie on the disjoint planes $z = \phi_i$ and $z = \phi_j$ of $\mathbb{R}^3$, respectively, since we assumed that the slopes $\phi_i$ and $\phi_j$ are distinct. Thus, $P_i \cap P_j = \emptyset$ (c.f. $P_2$ and $P_4$).

(c) One vertex is unbounded (that is, $t_i = \infty$) and the other is bounded (that is, $t_j \leq b_j - a_j$). If $\overline{P}_i \cap \overline{P}_j = \emptyset$, then $v_i v_j \notin E$ and $P_i \cap P_j = \emptyset$ (c.f. $P_1$ and $P_6$). Suppose that $\overline{P}_i \cap \overline{P}_j \neq \emptyset$. We distinguish two cases:

(i) $\phi_i < \phi_j$. It is easy to check that $|I_i \cap I_j| \geq t_j$ and thus $v_i v_j \in E$. Since $\overline{P}_i \cap \overline{P}_j \neq \emptyset$ and $\phi_i < \phi_j$, then necessarily the line segment $P_i$ intersects with the parallelepiped $P_j$ on the plane $z = \phi_i$, and thus $P_i \cap P_j \neq \emptyset$ (c.f. $P_1$ and $P_2$).

(ii) $\phi_i > \phi_j$. Clearly $|I_i \cap I_j| < t_i = \infty$. Furthermore, since $\phi_i > \phi_j$, Observation 1 implies that $|I_i \cap I_j| \leq |I_j| < t_i$. It follows that $|I_i \cap I_j| < \min\{t_i, t_j\}$, and thus $v_i v_j \notin E$. On the other hand, $z = \phi_i$ for all points $(x, y, z) \in P_i$, while $z \leq \phi_j < \phi_i$ for all points $(x, y, z) \in P_j$, and therefore $P_i \cap P_j = \emptyset$ (c.f. $P_3$ and $P_4$). □

Clearly, for each $v_i \in V$ the parallelepiped $P_i$ can be constructed in constant time. Therefore,

**Lemma 1.** *Given a tolerance representation of a tolerance graph $G$ with $n$ vertices, a parallelepiped representation of $G$ can be constructed in $\mathcal{O}(n)$ time.*

## 3 Coloring and Clique Algorithms in $\mathcal{O}(n \log n)$

In this section we present optimal $\mathcal{O}(n \log n)$ algorithms for constructing a minimum coloring and a maximum clique in a tolerance graph $G = (V, E)$ with $n$ vertices, given a parallelepiped representation of $G$. These algorithms improve the best known running time $\mathcal{O}(n^2)$ of these problems on tolerance graphs [12,13]. First, we introduce a canonical representation of tolerance graphs in Section 3.1, and then we use it to obtain the algorithms for the minimum coloring and the maximum clique problems in Section 3.2. Finally, we discuss the optimality of both algorithms in Section 3.4.

### 3.1 A canonical representation of tolerance graphs

We associate with every vertex $v_i$ of $G$ the point $p_i = (x_i, y_i)$ in the Euclidean plane, where $x_i = b_i$ and $y_i = \frac{\pi}{2} - \phi_i$. Since all endpoints of the parallelograms $\overline{P}_i$ and all slopes $\phi_i$ are distinct, all coordinates of the points $p_i$ are distinct as well. Similarly to [12, 13], we state the following two definitions.

**Definition 4.** *An unbounded vertex $v_i \in V_U$ of a tolerance graph $G$ is called* inevitable *(for a certain parallelepiped representation), if replacing $P_i$ with $\{(x, y, z) \mid (x, y) \in P_i, 0 \leq z \leq \phi_i\}$ creates a new edge in $G$. Otherwise, $v_i$ is called* evitable.

**Definition 5.** *Let $v_i \in V_U$ be an inevitable unbounded vertex of a tolerance graph $G$ (for a certain parallelepiped representation). A vertex $v_j$ is called a* hovering vertex *of $v_i$ if $a_j < a_i$, $b_i < b_j$, and $\phi_i > \phi_j$.*

It is now easy to see that, by Definition 5, if $v_j$ is a hovering vertex of $v_i$, then $v_i v_j \notin E$. Note that, in contrast to [12], in Definition 4, an isolated vertex $v_i$ might be also inevitable unbounded, while in Definition 5, a hovering vertex might be also unbounded. Definitions 4 and 5 imply the following lemma:

**Lemma 2.** *Let $v_i \in V_U$ be an inevitable unbounded vertex of the tolerance graph $G$ (for a certain parallelepiped representation). Then, there exists a hovering vertex $v_j$ of $v_i$.*

*Proof.* Since $v_i$ is an inevitable unbounded vertex, replacing $P_i$ with $\{(x, y, z) \mid (x, y) \in P_i, 0 \leq z \leq \phi_i\}$ creates a new edge in $G$; let $v_i v_j$ be such an edge. Then, clearly $\overline{P}_i \cap \overline{P}_j \neq \emptyset$. We will prove that $v_j$ is a hovering vertex of $v_i$. Otherwise, $\phi_i < \phi_j$, $a_j > a_i$, or $b_i > b_j$. Suppose first that $\phi_i < \phi_j$. If $v_j \in V_U$, then $v_i$ remains not connected to $v_j$ after the replacement of $P_i$ with $\{(x, y, z) \mid (x, y) \in P_i, 0 \leq z \leq \phi_i\}$, since $\phi_i < \phi_j$, which is a contradiction. If $v_j \in V_B$, then $v_i$ is connected to $v_j$ also before the replacement of $P_i$, since $\phi_i < \phi_j$ and $\overline{P}_i \cap \overline{P}_j \neq \emptyset$, which is again a contradiction. Thus, $\phi_i > \phi_j$. Suppose now that $a_j > a_i$ or $b_i > b_j$. Then, since $\phi_i > \phi_j$, we obtain for both cases that $\overline{P}_i \cap \overline{P}_j = \emptyset$, which is a contradiction. Thus, $a_j < a_i$, $b_i < b_j$, and $\phi_i > \phi_j$, i.e. $v_j$ is a hovering vertex of $v_i$ by Definition 5. □

**Definition 6.** *A parallelepiped representation of a tolerance graph $G$ is called* canonical *if every unbounded vertex is inevitable.*

For example, in the tolerance graph depicted in Figure 2, $v_4$ and $v_8$ are inevitable unbounded vertices, $v_3$ and $v_6$ are hovering vertices of $v_4$ and $v_8$, respectively, while $v_2$ and $v_6$ are evitable unbounded vertices. Therefore, this representation is not canonical for the graph $G$. However, if we replace $P_i$ with $\{(x, y, z) \mid (x, y) \in P_i, 0 \leq z \leq \phi_i\}$ for $i = 2, 6$, we get a canonical representation for $G$.

In the following, we present an algorithm that constructs a canonical representation of a given tolerance graph $G$.

**Definition 7.** *Let $\alpha = (x_\alpha, y_\alpha)$ and $\beta = (x_\beta, y_\beta)$ be two points in the plane. Then $\alpha$* dominates *$\beta$ if $x_\alpha > x_\beta$ and $y_\alpha > y_\beta$. Given a set $A$ of points, the point $\gamma \in A$ is called an* extreme point *of $A$ if there is no point $\delta \in A$ that dominates $\gamma$. $Ex(A)$ is the set of the extreme points of $A$.*

Given a tolerance graph $G = (V, E)$ with the set $V = \{v_1, v_2, \ldots, v_n\}$ of vertices (and its parallelepiped representation), we can assume w.l.o.g. that $a_i < a_j$ whenever $i < j$. Recall that with every vertex $v_i$ we associated the point $p_i = (x_i, y_i)$, where $x_i = b_i$ and $y_i = \frac{\pi}{2} - \phi_i$, respectively. We define for every $i = 1, 2, \ldots, n$ the set $A_i = \{p_1, p_2, \ldots, p_i\}$ of the points associated with the first $i$ vertices of $G$.

**Lemma 3.** *Let $v_i \in V_U$ be an unbounded vertex of a tolerance graph $G$. Then:*

*(a) If $p_i \in Ex(A_i)$ then $v_i$ is evitable.*
*(b) If $p_i \notin Ex(A_i)$ and point $p_j$ dominates $p_i$ for some bounded vertex $v_j \in V_B$ with $j < i$ then $v_i$ is inevitable and $v_j$ is a hovering vertex of $v_i$.*

*Proof.* (a) Assume, to the contrary, that $v_i$ is inevitable. By Lemma 2 there is a hovering vertex $v_j$ of $v_i$. But then, $x_i = b_i < b_j = x_j$ and $y_i = \frac{\pi}{2} - \phi_i < \frac{\pi}{2} - \phi_j = y_j$, while $a_j < a_i$, i.e. $j < i$. Therefore $p_j \in A_i$ and $p_j$ dominates $p_i$, which is a contradiction, since $p_i \in Ex(A_i)$.

(b) Suppose that $p_j$ dominates $p_i$, for some vertex $v_j \in V_B$ with $j < i$. The ordering of the vertices implies $a_j < a_i$, while $x_i < x_j$ and $y_i < y_j$ imply $b_i < b_j$ and $\phi_i > \phi_j$. Thus $v_i$ is inevitable and $v_j$ is a hovering vertex of $v_i$. □

The following theorem shows that, given a parallelepiped representation of a tolerance graph $G$, we can construct in $\mathcal{O}(n \log n)$ a canonical representation of $G$. This result is crucial for the time complexity analysis of the algorithms of Section 3.2.

**Theorem 2.** *Every parallelepiped representation of a tolerance graph $G$ with $n$ vertices can be transformed by Algorithm 1 to a canonical representation of $G$ in $\mathcal{O}(n \log n)$ time.*

*Proof.* We describe and analyze Algorithm 1 that generates a canonical representation of $G$. First, we sort the vertices $v_1, v_2, \ldots, v_n$ of $G$ such that $a_i < a_j$ whenever $i < j$. Then, we process sequentially all vertices $v_i$ of $G$. The bounded and the inevitable unbounded vertices will not be changed, while the evitable unbounded vertices will be replaced with bounded ones. At step $i$ we update the set $Ex(A_i)$ of the extreme points of $A_i$ (note that the set $A_i$ remains unchanged during the algorithm). For two points $p_{i_1}, p_{i_2}$ of $Ex(A_i)$, $x_{i_1} > x_{i_2}$ if and only if $y_{i_1} < y_{i_2}$. We store the elements of $Ex(A_i)$ in a list $P$, in which the points $p_j$ are sorted increasingly according to their $x$ values (or, equivalently, decreasingly according to their $y$ values). Due to Lemma 3(a), and since during the algorithm the evitable unbounded vertices of $G$ are replaced with bounded ones, after the process of vertex $v_i$, all points in the list $P$ correspond to bounded vertices of $G$ in the current parallelepiped representation.

We distinguish now the following cases:

**Algorithm 1** Construction of a canonical representation of a tolerance graph $G$

**Input:** A parallelepiped representation $R$ of a given tolerance graph $G$ with $n$ vertices
**Output:** A canonical representation $R'$ of $G$

```
Sort the vertices of G, such that a_i < a_j whenever i < j
ℓ_0 ← min{x_i : 1 ≤ i ≤ n}; r_0 ← max{x_i : 1 ≤ i ≤ n}
p_s ← (ℓ_0 − 1, π/2); p_t ← (r_0 + 1, 0)
P ← (p_s, p_t); R' ← R
for i = 1 to n do
   Find the point p_j having the smallest x_j with x_j > x_i
   if y_j < y_i then {no point of P dominates p_i}
      Find the point p_k having the greatest x_k with x_k < x_i
      Find the point p_ℓ having the greatest y_ℓ with y_ℓ < y_i
      if x_k ≥ x_ℓ then
         Replace points p_ℓ, p_{ℓ+1} ..., p_k with point p_i in the list P
      else
         Insert point p_i between points p_k and p_ℓ in the list P
      if v_i ∈ V_U then {v_i is an evitable unbounded vertex}
         Replace P_i with {(x, y, z) | (x, y) ∈ P_i, 0 ≤ z ≤ φ_i} in R'
   else {y_j > y_i; p_j dominates p_i}
      if v_i ∈ V_U then {v_i is an inevitable unbounded vertex}
         v_j is a hovering vertex of v_i
return R'
```

*Case 1.* $v_i$ is bounded. If there exists a point of $P$ that dominates $p_i$ then $p_i \notin Ex(A_i)$. Thus, we do not change $P$, and we continue to the process of $v_{i+1}$. If no point of $P$ dominates $p_i$ then $p_i \in Ex(A_i)$. Thus, we add $p_i$ to $P$ and we remove from $P$ all points that are dominated by $p_i$.

*Case 2.* $v_i$ is unbounded. If there exists a point $p_j \in P$ that dominates $p_i$ then $p_i \notin Ex(A_i)$, while Lemma 3(b) implies that $v_i$ is inevitable and $v_j$ is a hovering vertex of $v_i$. Thus, similarly to Case 1, we do not change $P$, and we continue to the process of $v_{i+1}$. If no point of $P$ dominates $p_i$ then $p_i \in Ex(A_i)$. Thus, we add the point $p_i$ to $P$ and remove from $P$ all points that are dominated by $p_i$. In this case, $v_i$ is evitable by Lemma 3(a). Hence, we replace $P_i$ with $\{(x, y, z) \mid (x, y) \in P_i, 0 \leq z \leq \phi_i\}$ in the current parallelepiped representation of $G$ and we consider from now on $v_i$ as a bounded vertex.

It follows that after the process of each vertex $v_i$ (either bounded or unbounded) the list $P$ stores the points of $Ex(A_i)$. Furthermore, at every iteration of the algorithm, all points of the list $P$ correspond to bounded vertices in the current parallelepiped representation of $G$.

The processing of vertex $v_i$ is done by executing three binary searches in the list $P$ as follows. Let $\ell_0 = \min\{x_i \mid 1 \leq i \leq n\}$ and $r_0 = \max\{x_i \mid 1 \leq i \leq n\}$. For convenience, we add two dummy points $p_s = (\ell_0 - 1, \frac{\pi}{2})$ and $p_t = (r_0 + 1, 0)$. First, we find the point $p_j \in P$ with the smallest value $x_j$, such that $x_j > x_i$ (see Figure 3). Note that $p_i \in Ex(A_i)$ if and only if $y_j < y_i$. If $y_j > y_i$ then $p_j$ dominates $p_i$ (see Figure 3(a)). Thus, if $v_i \in V_U$, Lemma 3(b) implies that $v_i$ is an inevitable unbounded vertex and $v_j$ is a hovering vertex of $v_i$. In the opposite case $y_j < y_i$, we have to add $p_i$ to $P$. In order to remove from $P$ all points that are dominated by $p_i$, we execute binary search two more times. In particular, we find the points $p_k$ and $p_\ell$ of $P$ with the greatest values $x_k$ and $y_\ell$, respectively, such that $x_k < x_i$ and $y_\ell < y_i$ (see Figure 3(b)). If there are some points of $P$ that are dominated by $p_i$, then $p_k$ and $p_\ell$ have the greatest and smallest values $x_k$ and $x_\ell$ among them, respectively, and $x_k \geq x_\ell$. In this case, we replace all points $p_\ell, p_{\ell+1}, \ldots, p_k$ with the point $p_i$ in the list $P$. Otherwise, if no point of $P$ is dominated by $p_i$, then $x_k < x_\ell$. In this case, we remove no point from $P$ and we insert $p_i$ between $p_k$ and $p_\ell$ in $P$.

Finally, after processing all vertices $v_i$ of $G$, we return a canonical representation of the given tolerance graph $G$, in which every vertex that remains unbounded has a hovering vertex

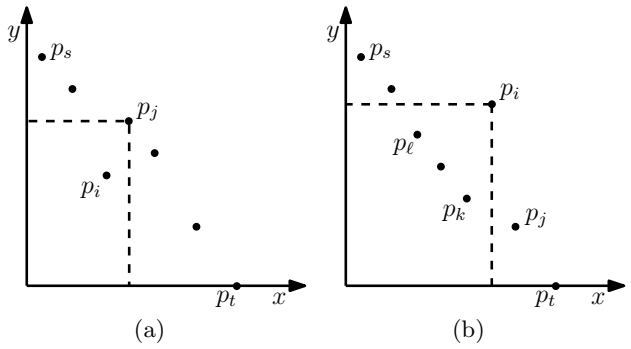


**Fig. 3.** The cases where the associated point $p_i$ to the currently processed vertex $v_i$ is (a) dominated by the point $p_j$ in $A_i$ and (b) an extreme point of the set $A_i$.

assigned to it. Since the processing of every vertex can be done in $\mathcal{O}(\log n)$ time by executing three binary searches, and since the sorting of the vertices can be done in $\mathcal{O}(n \log n)$ time, the running time of Algorithm 1 is $\mathcal{O}(n \log n)$. □

### 3.2 Minimum coloring

In the next theorem we present an optimal $\mathcal{O}(n \log n)$ algorithm for computing a minimum coloring of a tolerance graph $G$ with $n$ vertices, given a parallelepiped representation of $G$. The informal description of the algorithm is identical to the one in [12], which has running time $\mathcal{O}(n^2)$; the difference is in the fact that we use our new representation, in order to improve the time complexity.

**Algorithm 2** Minimum coloring of a tolerance graph $G$

**Input:** A parallelepiped representation of a given tolerance graph $G$
**Output:** A minimum coloring of $G$

Construct a canonical representation of $G$ by Algorithm 1, where a hovering vertex is associated with every inevitable unbounded vertex
Color $G[V_B]$ by the algorithm of [6]
**for** every inevitable unbounded vertex $v_i \in V_U$ **do**
  Assign to $v_i$ the same color as its hovering vertex in $G[V_B]$

**Theorem 3.** *A minimum coloring of a tolerance graph $G$ with $n$ vertices can be computed in $\mathcal{O}(n \log n)$ time.*

*Proof.* We present Algorithm 2 that computes a minimum coloring of $G$. Given a parallelepiped representation of $G$, we construct a canonical representation of $G$ in $\mathcal{O}(n \log n)$ time by Algorithm 1. $V_B$ and $V_U$ are the sets of bounded and inevitable unbounded vertices of $G$ in the latter representation, respectively. In particular, Algorithm 1 associates a hovering vertex $v_j \in V_B$ with every inevitable unbounded vertex $v_i \in V_U$. We find a minimum proper coloring of the bounded tolerance graph $G[V_B]$ in $\mathcal{O}(n \log n)$ time using the algorithm of [6]. Finally, we associate with every inevitable unbounded vertex $v_i \in V_U$ the same color as that of its hovering vertex $v_j \in V_B$ in the coloring of $G[V_B]$.

Consider an arbitrary inevitable unbounded vertex $v_i \in V_U$ and its hovering vertex $v_j \in V_B$. Following Definition 5, $\overline{P}_i \cap \overline{P}_j \neq \emptyset$ and $\phi_i > \phi_j$. Consider a vertex $v_k$ of $G$, such that $v_i v_k \in E$. It follows that $v_k \in V_B$, since no two unbounded vertices are adjacent in $G$. Furthermore, since

$v_iv_k \in E$, it follows that $\overline{P}_i \cap \overline{P}_k \neq \emptyset$ and $\phi_k > \phi_i$. Then $\overline{P}_j \cap \overline{P}_k \neq \emptyset$, and thus $P_j \cap P_k \neq \emptyset$, i.e. $v_jv_k \in E$, since both $v_j$ and $v_k$ are bounded vertices. It follows that $v_k$ does not have the same color as $v_j$ in the proper coloring of $G[V_B]$, and thus the resulting coloring of $G$ is proper. Finally, since both colorings of $G[V_B]$ and of $G$ have the same number of colors, it follows that this proper coloring of $G$ is minimum. Since the coloring of $G[V_B]$ can be done in $\mathcal{O}(n \log n)$ time and the coloring of all inevitable unbounded vertices $v_i \in V_U$ can be done in $\mathcal{O}(n)$ time, Algorithm 2 returns a minimum proper coloring $G$ in $\mathcal{O}(n \log n)$ time. □

### 3.3 Maximum clique

In the next theorem we prove that a maximum clique of a tolerance graph $G$ with $n$ vertices can be computed in optimal $\mathcal{O}(n \log n)$ time, given a parallelepiped representation of $G$. This theorem follows from Theorem 2 and from the clique algorithm presented in [6], and it improves the best known $\mathcal{O}(n^2)$ running time mentioned in [12].

**Theorem 4.** *A maximum clique of a tolerance graph $G$ with $n$ vertices can be computed in $\mathcal{O}(n \log n)$ time.*

*Proof.* We compute first a canonical representation of $G$ in $\mathcal{O}(n \log n)$ time by Algorithm 1. The proof of Theorem 3 implies that $\chi(G) = \chi(G[V_B])$, where $\chi(H)$ denotes the chromaric number of a given graph $H$. Since tolerance graphs are perfect graphs [11], $\omega(G) = \chi(G)$ and $\omega(G[V_B]) = \chi(G[V_B])$, where $\omega(H)$ denotes the clique number of a given graph $H$. It follows that $\omega(G) = \omega(G[V_B])$. We compute now a maximum clique $Q$ of the bounded tolerance graph $G[V_B]$ in $\mathcal{O}(n \log n)$ time. This can be done by the algorithm presented in [6] that computes a maximum clique in a trapezoid graph, since bounded tolerance graphs are trapezoid graphs [13]. Since $\omega(G) = \omega(G[V_B])$, $Q$ is a maximum clique of $G$ as well. □

### 3.4 Optimality of the running time

In this section we use permutation graphs [13]. Given a sequence $S = a_1, a_2, \ldots, a_n$ of numbers, a *subsequence* of $S$ is a sequence $S' = a_{i_1}, a_{i_2}, \ldots, a_{i_k}$, where $a_{i_j} \in S$ for every $j \in \{1, 2, \ldots, k\}$, and $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. $S'$ is called an *increasing subsequence* of $S$, if $a_{i_1} < a_{i_2} < \ldots < a_{i_k}$. Clearly, increasing subsequences in a permutation graph $G$ correspond to independent sets of $G$, while increasing subsequences in the complement $\overline{G}$ of $G$ correspond to cliques of $G$, where $\overline{G}$ is also a permutation graph. Since $\Omega(n \log n)$ is a lower time bound for computing the length of a longest increasing subsequence in a permutation [6, 8], the same lower time bound holds for computing a maximum clique and a maximum independent set in a permutation graph $G$. Furthermore, since permutation graphs are perfect graphs [9], the chromatic number $\chi(G)$ of a permutation graph $G$ equals the clique number $\omega(G)$ of $G$. Thus, $\Omega(n \log n)$ is a lower time bound for computing the chromatic number of a permutation graph. Finally, since the class of permutation graphs is a subclass of tolerance graphs [13], the same lower bounds hold for tolerance graphs. It follows that the algorithms in Theorems 3 and 4 for computing a minimum coloring and a maximum clique in tolerance graphs are optimal.

## 4 Weighted Independent Set Algorithm in $\mathcal{O}(n^2)$

In this section we present an algorithm for computing a maximum weight independent set in a tolerance graph $G = (V, E)$ with $n$ vertices in $\mathcal{O}(n^2)$ time, given a parallelepiped representation of $G$, and a weight $w(v_i) > 0$ for every vertex $v_i$ of $G$. The proposed algorithm improves the running time $\mathcal{O}(n^3)$ of the one presented in [13]. In the following, consider as above the

partition of the vertex set $V$ into the sets $V_B$ and $V_U$ of bounded and unbounded vertices of $G$, respectively.

Similarly to [13], we add two isolated bounded vertices $v_s$ and $v_t$ to $G$ with weights $w(v_s) = w(v_t) = 0$, such that the corresponding parallelepipeds $P_s$ and $P_t$ lie completely to the left and to the right of all other parallelepipeds of $G$, respectively. Since both $v_s$ and $v_t$ are bounded vertices, we augment the set $V_B$ by the vertices $v_s$ and $v_t$. In particular, we define the set of vertices $V'_B = V_B \cup \{v_s, v_t\}$ and the tolerance graph $G' = (V', E)$, where $V' = V'_B \cup V_U$. Since $G'[V'_B]$ is a bounded tolerance graph, it is a co-comparability graph as well [11,13]. A transitive orientation of the comparability graph $\overline{G'[V'_B]}$ can be obtained by directing each edge according to the upper left endpoints of the parallelograms $\overline{P}_i$. Formally, let $(V'_B, \prec)$ be the partial order defined on the bounded vertices $V'_B$, such that $v_i \prec v_j$ if and only if $v_iv_j \notin E$ and $c_i < c_j$. Recall that a *chain* of elements in a partial order is a set of mutually comparable elements in this order [4].

**Observation 2 ([13])** *The independent sets of $G[V_B]$ are in one-to-one correspondence with the chains in the partial order $(V'_B, \prec)$ from $v_s$ to $v_t$.*

For the sequel, recall that for every unbounded vertex $v_k \in V_U$ the parallelepiped $P_k$ degenerates to a line segment, while the upper endpoints $b_k$ and $c_k$ of the parallelogram $\overline{P}_k$ coincide, i.e. $b_k = c_k$.

**Definition 8.** *For every $v_i, v_j \in V'_B$ with $v_i \prec v_j$, $L_i(j) = \{v_k \in V_U \mid b_i < c_k < c_j, v_iv_k \notin E\}$ and its weight $w(L_i(j)) = \sum_{v \in L_i(j)} w(v)$.*

**Definition 9.** *For every $v_j \in V'_B$, $R_j = \{v_k \in V_U \mid c_j < c_k < b_j, v_jv_k \notin E\}$ and its weight $w(R_j) = \sum_{v \in R_j} w(v)$.*

For every pair of bounded vertices $v_i, v_j \in V'_B$ with $v_i \prec v_j$, the set $L_i(j)$ consists of those unbounded vertices $v_k \in V_U$, for which $v_iv_k \notin E$ and whose upper endpoint $b_k = c_k$ of $\overline{P}_k$ lies between $\overline{P}_i$ and $\overline{P}_j$. Furthermore, $v_jv_k \notin E$ for every vertex $v_k \in L_i(j)$. Indeed, in the case where $\overline{P}_k \cap \overline{P}_j \neq \emptyset$, it holds $\phi_k > \phi_j$, since $b_k = c_k < c_j$, and thus $P_k \cap P_j = \emptyset$. Similarly, the set $R_j$ consists of those unbounded vertices $v_k \in V_U$, for which $v_jv_k \notin E$ and whose upper endpoint $b_k = c_k$ of $\overline{P}_k$ lies between the upper endpoints $c_j$ and $b_j$ of $\overline{P}_j$. Furthermore, $v_iv_k \notin E$ for every vertex $v_k \in R_j$ as well. Indeed, since $v_jv_k \notin E$, it follows that $\phi_k > \phi_j$, and thus, $\overline{P}_i \cap \overline{P}_k = \emptyset$ and $P_i \cap P_k = \emptyset$. In particular, in the example of Figure 4, $L_1(2) = \{v_3, v_5\}$ and $R_2 = \{v_6\}$. In this figure, the line segments that correspond to the unbounded vertices $v_4$ and $v_7$, respectively, are drawn with dotted lines to illustrate the fact that $v_4v_1 \in E$ and $v_7v_2 \in E$.

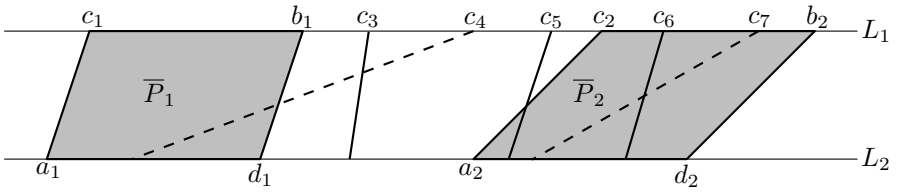


**Fig. 4.** The parallelograms $\overline{P}_i$, $i = 1, 2, \ldots, 7$ of a tolerance graph with the sets $V_B = \{v_1, v_2\}$ and $V_U = \{v_3, v_4, \ldots, v_7\}$ of bounded and unbounded vertices, respectively. In this graph, $L_1(2) = \{v_3, v_5\}$, $R_2 = \{v_6\}$ and $S(v_1, v_2) = \{v_3, v_5, v_6\}$.

**Definition 10 ([13]).** *For every $v_i, v_j \in V'_B$ with $v_i \prec v_j$, $S(v_i, v_j) = \{v_k \in V_U \mid v_iv_k, v_jv_k \notin E, b_i < c_k < b_j\}$.*

**Algorithm 3** Maximum weight independent set of a tolerance graph $G$

**Input:** A parallelepiped representation of a given tolerance graph $G$
**Output:** The value of a maximum weight independent set of $G$

Add the dummy bounded vertices $v_s, v_t$ to $G$, such that $P_s$ and $P_t$ lie completely to the left and to the right of all other parallelepipeds of $G$, respectively
$V'_B \leftarrow V_B \cup \{v_s, v_t\}$
Construct the partial ordering $(V'_B, \prec)$ of the bounded vertices $V'_B$
Sort the bounded vertices $V'_B$, such that $c_i < c_j$ whenever $i < j$

**for** $j = 1$ to $|V'_B|$ **do**
  $W(v_j) \leftarrow 0$
  Compute the value $w(R_j)$

**for** $i = 1$ to $|V'_B|$ **do**
  **for** every $v_j \in V'_B$ with $v_i \prec v_j$ **do**
    Update the value $w(L_i(j))$
    **if** $W(v_j) < (w(v_j) + w(R_j)) + W(v_i) + w(L_i(j))$ **then**
      $W(v_j) \leftarrow (w(v_j) + w(R_j)) + W(v_i) + w(L_i(j))$
**return** $W(v_t)$

**Observation 3** *For every pair of bounded vertices $v_i, v_j \in V'_B$ with $v_i \prec v_j$,*

$$S(v_i, v_j) = L_i(j) \cup R_j \tag{1}$$

*Furthermore, $L_i(j) \subseteq L_i(\ell)$ for every triple $\{v_i, v_j, v_\ell\}$ of bounded vertices, where $v_i \prec v_j$, $v_i \prec v_\ell$ and $c_j < c_\ell$.*

In particular, in the example of Figure 4, $S(v_1, v_2) = L_1(2) \cup R_2 = \{v_3, v_5, v_6\}$.

**Lemma 4 ([13]).** *Given a tolerance graph $G$ with a set of positive weights for the vertices of $G$, any maximum weight independent set of $G$ consists of a chain of bounded vertices $v_{x_1} \prec v_{x_2} \prec \ldots \prec v_{x_k}$ together with the union of the sets $\cup\{S(v_{x_i}, v_{x_{i+1}}) \mid i = 0, 1, \ldots, k\}$, where $v_{x_0} = v_s$ and $v_{x_{k+1}} = v_t$.*

Now, using Lemma 4 and Observation 3, we can present Algorithm 3, which improves the running time $\mathcal{O}(n^3)$ of the one presented in [13].

**Theorem 5.** *A maximum weight independent set of a tolerance graph $G$ with $n$ vertices can be computed using Algorithm 3 in $\mathcal{O}(n^2)$ time.*

*Proof.* We present Algorithm 3 that computes the value of a maximum weight independent set of $G$. A slight modification of Algorithm 3 returns a maximum weight independent set of $G$, instead of its value. First, we construct the partial order $(V'_B, \prec)$ defined on the bounded vertices $V'_B = V_B \cup \{v_s, v_t\}$, such that $v_i \prec v_j$ whenever $v_i v_j \notin E$ and $c_i < c_j$. This can be done in $\mathcal{O}(n^2)$ time. Then, we sort the bounded vertices of $V'_B$, such that $c_i < c_j$ whenever $i < j$. This can be done in $\mathcal{O}(n \log n)$ time. As a preprocessing step, we compute for every bounded vertex $v_j \in V'_B$ the set $R_j$ and its weight $w(R_j)$ in linear $\mathcal{O}(n)$ time by visiting at most all unbounded vertices $v_k \in V_U$. Thus, all values $w(R_j)$ are computed in $\mathcal{O}(n^2)$ time.

We associate with each bounded vertex $v_j \in V'_B$ a cumulative weight $W(v_j)$ defined as follows:

$$\begin{aligned} &W(v_s) = 0 \\ &W(v_j) = (w(v_j) + w(R_j)) + \max_{v_i \prec v_j}\{W(v_i) + w(L_i(j))\}, \text{ for every } v_j \in V'_B \setminus \{v_s\} \end{aligned} \tag{2}$$

The cumulative weight $W(v_j)$ of an arbitrary bounded vertex $v_j \in V'_B$ equals the maximum weight of an independent set $S$ of vertices $v_k$ (both bounded and unbounded), for which $b_k \leq b_j$ and $v_j \in S$. Initially all values $W(v_j)$ are set to zero.

In the main part of Algorithm 3, we process sequentially all bounded vertices $v_i \in V'_B$. For every such vertex $v_i$, we update sequentially the cumulative weights $W(v_j)$ for all bounded vertices $v_j \in V'_B$ with $v_i \prec v_j$ by comparing the current value of $W(v_j)$ with the value $(w(v_j) + w(R_j)) + W(v_i) + w(L_i(j))$, and by storing the greatest of them in $W(v_j)$. After all bounded vertices of $V'_B$ have been processed, the value of the maximum weight independent set of $G$ is stored in $W(v_t)$, due to Lemma 4 and Observation 3.

While processing the bounded vertex $v_i$, we compute the values $w(L_i(j))$ sequentially for every $j$, where $v_i \prec v_j$, as follows. Let $v_{j_1}, v_{j_2}$ be two bounded vertices that are visited consecutively by the algorithm, during the process of vertex $v_i$. Then, due to Observation 3, we compute the value $w(L_i(j_2))$ by adding to the previous value $w(L_i(j_1))$ the weights of all unbounded vertices $v_k \in V_U$, whose upper endpoints $c_k$ lie between $c_{j_1}$ and $c_{j_2}$.

Since we visit all bounded and all unbounded vertices of the graph at most once during the process of $v_i$, this can be done in $\mathcal{O}(n)$ time. Thus, since there are in total at most $n+2$ bounded vertices $v_i \in V'_B$, Algorithm 3 returns the value of the maximum weight independent set of $G$ in $\mathcal{O}(n^2)$ time. Finally, observe that, storing at every step of Algorithm 3 the independent sets that correspond to the values $W(v_i)$, and removing at the end the vertices $v_s$ and $v_t$, the algorithm returns at the same time a maximum weight independent set of $G$, instead of its value. □

## 5 Conclusions and Further Research

In this article we proposed the first non-trivial intersection model for general tolerance graphs, given by parallelepipeds in the three-dimensional space. This representation generalizes the parallelogram representation of bounded tolerance graphs. Using this representation, we presented improved algorithms for computing a minimum coloring, a maximum clique, and a maximum weight independent set on a tolerance graph. The running times of the first two algorithms are optimal. It can be expected that this representation will prove useful in improving the running time of other algorithms for the class of tolerance graphs.

As mentioned in Section 1, the complexity of the recognition problem for tolerance and bounded tolerance graphs is possibly the main open problem in this class of graphs. Even when the input graph is known to be a tolerance graph, it is not known how to obtain a tolerance representation for it [22]. Moreover, given a tolerance graph, it is not known how to decide in polynomial time whether it is a bounded tolerance graph [22].

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